arXiv:math/0512563v2 [math.QA] 19 Sep 2007

QUANTUM DOUBLE OF $\mathrm{U}_q((\mathfrak{sl}_2)^{\leq 0})$

JUN HU AND YINHUO ZHANG

Abstract. Let $\mathrm{U}_q(\mathfrak{sl}_2)$ be the quantized enveloping algebra associated to the simple Lie algebra $\mathfrak{sl}_2$. In this paper, we study the quantum double D_q of the Borel subalgebra $\mathrm{U}_q((\mathfrak{sl}_2)^{\leq 0})$ of $\mathrm{U}_q(\mathfrak{sl}_2)$. We construct an analogue of Kostant–Lusztig $\mathbb{Z}[v, v^{-1}]$-form for D_q and show that it is a Hopf subalgebra. We prove that, over an algebraically closed field, every simple D_q-module is the pullback of a simple $\mathrm{U}_q(\mathfrak{sl}_2)$-module through certain surjection from D_q onto $\mathrm{U}_q(\mathfrak{sl}_2)$, and the category of finite dimensional weight D_q-modules is equivalent to a direct sum of $|k^\times|$ copies of the category of finite dimensional weight $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. As an application, we recover (in a conceptual way) Chen's results [2] as well as Radford's results [20] on the quantum double of Taft algebra. Our main results allow a direct generalization to the quantum double of the Borel subalgebra of the quantized enveloping algebra associated to arbitrary Cartan matrix.

1. Preliminaries

Let k be a field. Let q be an invertible element in k satisfying $q^2 \neq 1$. The quantized enveloping algebra[1] associated to the simple Lie algebra $\mathfrak{sl}_2$ is the associative k-algebra with the generators E, F, K, K^{-1} and the relations (cf. [6] and [9], [10]):

$$KE = q^2 EK,\ KF = q^{-2} FK,\ KK^{-1} = 1 = K^{-1}K,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We denote it by $\mathrm{U}_q(\mathfrak{sl}_2)$ or just U_q for simplicity. The algebra U_q is a quantum analogue of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ associated to the simple Lie algebra $\mathfrak{sl}_2$. It is a Hopf algebra with comultiplication, counit and antipode given by:

$$\Delta(E) = E \otimes 1 + K \otimes E,\ \Delta(F) = F \otimes K^{-1} + 1 \otimes F,\ \Delta(K) = K \otimes K,$$
$$\varepsilon(E) = 0 = \varepsilon(F),\ \varepsilon(K) = 1 = \varepsilon(K^{-1}),$$
$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1}.$$

1991 *Mathematics Subject Classification.* 16W30.

Key words and phrases. Hopf algebra, Drinfel'd double, quantized enveloping algebra.

[1] We are actually working with De Concini–Kac's version of specialized quantum algebra, see [4].

Let U_q^+ (resp. U_q^-) be the k-subalgebra of U_q generated by E (resp. by F). Let U_q^0 be the k-subalgebra of U_q generated by K, K^{-1}. Then the elements $\{E^a\}$ (resp. $\{F^b\}$), where $a, b \in \mathbb{N} \cup \{0\}$, form a k-basis of U_q^+ (resp. of U_q^-). The elements $\{K^c\}$, where $c \in \mathbb{Z}$, form a k-basis of U_q^0. Moreover, the natural k-linear map $\mathrm{U}_q^+ \otimes \mathrm{U}_q^0 \otimes \mathrm{U}_q^- \to \mathrm{U}_q$ given by multiplication is a k-linear isomorphism. The basis $\left\{E^a K^c F^b \;\middle|\; a, b \in \mathbb{N} \cup \{0\}, c \in \mathbb{Z}\right\}$ is called the PBW basis of U_q. We define $\mathrm{U}_q^{\geq 0} := \mathrm{U}_q^+ \mathrm{U}_q^0$, $\mathrm{U}_q^{\leq 0} := \mathrm{U}_q^- \mathrm{U}_q^0$. Then both $\mathrm{U}_q^{\geq 0}$ and $\mathrm{U}_q^{\leq 0}$ are Hopf k-subalgebras of U_q. For any monomials $E^a K^b, F^c K^d$, we endow them *the degree* a, c respectively.

Let v be an indeterminate over $\mathbb{Z}$. We consider the quantized enveloping algebra $\mathrm{U}_v = \mathrm{U}_v(\mathfrak{sl}_2)$ with parameter v and defined over $\mathbb{Q}(v)$. It is well-known (see [11], [17] and [25][2]) that there exists a unique pairing $\varphi : \mathrm{U}_v^{\geq 0} \times \mathrm{U}_v^{\leq 0} \to \mathbb{Q}(v)$ such that

(1) $\varphi(1,1) = 1,\ \varphi(1, K) = 1 = \varphi(K, 1)$

(2) $\varphi(x, y) = 0,$ if x, y are homogeneous with different degree,

(3) $\varphi(E, F) = \dfrac{1}{v^2 - 1},\ \varphi(K, K) = v^2,\ \varphi(K, K^{-1}) = v^{-2},$

(4) $\varphi(x, y'y'') = \varphi(\Delta^{\mathrm{op}}(x), y' \otimes y''),$ for all $x \in \mathrm{U}_v^{\geq 0}$, $y', y'' \in \mathrm{U}_v^{\leq 0}$,

(5) $\varphi(xx', y'') = \varphi(x \otimes x', \Delta(y'')),$ for all $x, x' \in \mathrm{U}_v^{\geq 0}$, $y'' \in \mathrm{U}_v^{\leq 0}$,

(6) $\varphi(S(x), y) = \varphi(x, S^{-1}(y)),$ for all $x \in \mathrm{U}_v^{\geq 0}$, $y \in \mathrm{U}_v^{\leq 0}$.

One usually call $(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0}, \varphi)$ a skew Hopf pairing (cf. [18]). Then, one can make $D(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0}) := \mathrm{U}_v^{\geq 0} \otimes \mathrm{U}_v^{\leq 0}$ into a Hopf $\mathbb{Q}(v)$-algebra, which is called the quantum double of $(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0}, \varphi)$. As a $\mathbb{Q}(v)$-coalgebra, $D(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0}) = \mathrm{U}_v^{\geq 0} \otimes \mathrm{U}_v^{\leq 0}$, the tensor product of two coalgebras. The algebra structure of $D(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0})$ is determined by

$$(7)\quad (x \otimes y)(x' \otimes y') = \sum \varphi\big(x'_{(1)}, y_{(1)}\big)(xx'_{(2)} \otimes y_{(2)} y')\varphi\big(x'_{(3)}, S^{-1}(y_{(3)})\big),$$

for all $x, x' \in \mathrm{U}_v^{\geq 0}, y, y' \in \mathrm{U}_v^{\leq 0}$, where Sweedler's sigma summation $\Delta^2(y) = \sum y_{(1)} \otimes y_{(2)} \otimes y_{(3)}$ is used. Note that the quantum double we described here actually arises from a 2-cocycle twist (see [5] for more detail). The representation results in this paper are related to the recent work of Radford and Schneider (see [21] and [22]). We thank the referee for pointing out these references. For simplicity, we shall write D_v instead of $D(\mathrm{U}_v^{\geq 0}, \mathrm{U}_v^{\leq 0})$.

Let $A = \mathbb{Z}[v, v^{-1}, (v - v^{-1})^{-1}]$. Let $\mathrm{U}_{A,v}^{\leq 0}$ be the associative A-algebra defined by the generators F, K, K^{-1} and relations

$$KF = v^{-2} FK,\ KK^{-1} = 1 = K^{-1}K.$$

[2]Note that we use a slightly different version here so that the multiplication rule is compatible with the one given in [12, Chapter IX, (4.3)] for the Drinfel'd quantum double of finite dimensional Hopf algebras.

Specializing v to q, we make k into an A-algebra. Clearly $\mathrm{U}_q^{\leq 0} \cong k \otimes_A \mathrm{U}_{A,v}^{\leq 0}$. Similarly, we can define $\mathrm{U}_{A,v}^{\geq 0}$, and we have $\mathrm{U}_q^{\geq 0} \cong k \otimes_A \mathrm{U}_{A,v}^{\geq 0}$.

The previous construction of skew Hopf pairing clearly gives rise to a pairing $\varphi : \mathrm{U}_{A,v}^{\geq 0} \times \mathrm{U}_{A,v}^{\leq 0} \to A$, and hence gives rise to a pairing $\varphi : \mathrm{U}_q^{\geq 0} \times \mathrm{U}_q^{\leq 0} \to k$.

Lemma 1.1. *The pairing φ gives rise to a Hopf algebra map θ from the Hopf algebra $\mathrm{U}_q^{\leq 0}$ to the Hopf algebra $\left(\mathrm{U}_q^{\geq 0}\right)^{*,\mathrm{op}}$ as well as a Hopf algebra map θ' from the Hopf algebra $\mathrm{U}_q^{\geq 0}$ to the Hopf algebra $\left(\mathrm{U}_q^{\leq 0}\right)^{*,\mathrm{cop}}$. Moreover, θ, θ' are injective if q is not a root of unity.*

Proof. The maps θ, θ' are defined by

$$\theta(y)(x) = \varphi(x, y),$$
$$\theta'(x)(y) = \varphi(x, y),$$

for any $x \in \mathrm{U}_q^{\geq 0}, y \in \mathrm{U}_q^{\leq 0}$.

Now the first statement follows directly from the definition of φ. The second statement can be proved by using a similar argument in the proof of Lemma 4.1. □

Now we can construct the quantum double $D(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0}) := \mathrm{U}_q^{\geq 0} \otimes \mathrm{U}_q^{\leq 0}$ of $(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0}, \varphi)$ in a similar way, making it into a Hopf k-algebra. Henceforth, we shall write D_q instead of $D(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0})$. We have the following.

Theorem 1.2. *As a k-algebra, D_q can be presented by the generators*

$$E, F, K, K^{-1}, \widetilde{K}, \widetilde{K}^{-1},$$

and the following relations:

$$KE = q^2 EK,\ KF = q^{-2} FK,\ \widetilde{K}E = q^2 E\widetilde{K},\ \widetilde{K}F = q^{-2} F\widetilde{K},$$
$$KK^{-1} = K^{-1}K = 1 = \widetilde{K}\widetilde{K}^{-1} = \widetilde{K}^{-1}\widetilde{K},\ K\widetilde{K} = \widetilde{K}K,$$
$$EF - FE = \frac{K - \widetilde{K}^{-1}}{q - q^{-1}}.$$

Proof. Let D_q' be an abstract k-algebra defined by the generators and relations as above. One checks directly that D_q is generated by the following elements

$$E \otimes 1,\ 1 \otimes qF,\ K^{\pm 1} \otimes 1,\ 1 \otimes K^{\pm 1},$$

and these elements satisfy the above relations. In other words, there is a surjective k-algebra homomorphism $\psi : \mathrm{D}_q' \to \mathrm{D}_q$ such that

$$\psi(E) = E \otimes 1,\ \psi(F) = 1 \otimes qF,\ \psi(K^{\pm 1}) = K^{\pm 1} \otimes 1,\ \psi(\widetilde{K}^{\pm 1}) = 1 \otimes K^{\pm 1}.$$

On the other hand, we claim that the monomials

$$E^a K^c \widetilde{K}^d F^b, \quad a, b, c, d \in \mathbb{Z},\ a, b \geq 0,$$

form a basis of D_q'. We prove this by using a similar argument as in the proof of [8, Theorem 1.5]. We consider a polynomial ring $k[T_1, T_2, T_3, T_4]$ in four indeterminate T_1, T_2, T_3, T_4 and its localization $A' = k[T_1, T_2, T_2^{-1}, T_3, T_3^{-1}, T_4]$. Then all monomials $T_1^a T_2^c T_3^d T_4^b$ with $a, b, c, d \in \mathbb{Z}, a, b \geq 0$ are a basis of A'. We define linear endomorphisms $e, f, h, \widetilde{h}$ of A' by letting

$$
\begin{aligned}
e\big(T_1^a T_2^c T_3^d T_4^b\big) &= T_1^{a+1} T_2^c T_3^d T_4^b, \\
f\big(T_1^a T_2^c T_3^d T_4^b\big) &= \begin{cases} -T_1^{a-1}[a-1]_q \dfrac{q^{a-1}K - q^{1-a}\widetilde{K}^{-1}}{q-q^{-1}} T_2^c T_3^d T_4^b & \text{if } a \geq 1; \\ \qquad\qquad +q^{2c+2d} T_1^a T_2^c T_3^d T_4^{b+1}, & \\ q^{2c+2d} T_2^c T_3^d T_4^{b+1}, & \text{if } a = 0, \end{cases} \\
h\big(T_1^a T_2^c T_3^d T_4^b\big) &= q^{2a} T_1^a T_2^{c+1} T_3^d T_4^b, \\
\widetilde{h}\big(T_1^a T_2^c T_3^d T_4^b\big) &= q^{2a} T_1^a T_2^c T_3^{d+1} T_4^b,
\end{aligned}
$$

where

$$[a-1]_q := \frac{q^{a-1} - q^{1-a}}{q - q^{-1}}.$$

One can check that the above definition gives rise to a representation of D_q' on A' by taking E to e, F to f, $K^{\pm 1}$ to $h^{\pm 1}$ and $\widetilde{K}^{\pm 1}$ to $\widetilde{h}^{\pm 1}$. So it takes a monomial $E^a K^c \widetilde{K}^d F^b$ to the monomial $e^a h^c \widetilde{h}^d f^b$. Note that

$$e^a h^c \widetilde{h}^d f^b(1) = T_1^a T_2^c T_3^d T_4^b,$$

which implies that the $e^a h^c \widetilde{h}^d f^b$ are linearly independent, hence the monomials $E^a K^c \widetilde{K}^d F^b$ must be linear independent as well. This proves our claim. By definition of D_q, we know that the monomials

$$(E \otimes 1)^a (K \otimes 1)^c (1 \otimes \widetilde{K})^d (1 \otimes F)^b, \quad a, b, c, d \in \mathbb{Z},\ a, b \geq 0,$$

are a basis of D_q. Therefore, ψ maps a basis of D_q' onto a basis of D_q. It follows that ψ must be an isomorphism, as required. □

Lemma 1.3. *The map which sends E to E, F to F, $K^{\pm 1}$ to $K^{\pm 1}$ and $\widetilde{K}^{\pm 1}$ to $K^{\pm 1}$ extends uniquely to a surjective Hopf algebra homomorphism $\pi : \mathrm{D}_q \twoheadrightarrow \mathrm{U}_q$.*

Proof. This is obvious. Note that the kernel of π is the two-sided ideal of D_q generated by $K - \widetilde{K}$, which is in fact a Hopf ideal of D_q. □

The algebra D_q will be the primary interest to us in this paper. It turns out that this algebra behaves quite similar to the quantized enveloping algebra U_q in many ways. In the following sections we shall see that many constructions and equalities in the structure and representation theory of U_q carry over to the algebra D_q.

2. An analogue of Kostant–Lusztig $\mathbb{Z}[v, v^{-1}]$-form

The purpose of this section is to construct an analogue of Kostant–Lusztig $\mathbb{Z}[v, v^{-1}]$-form for the quantum double D_q.

Let R be an integral domain. Let v be an indeterminate over R. Let $\mathcal{A} = R[v, v^{-1}]$, the ring of Laurent R-polynomials in v. We consider the quantum double D_v (with parameter v) defined over the quotient field of $\mathcal{A}$. We shall define an analogue of Kostant–Lusztig $\mathcal{A}$-form (see [13], [15], [16]) for the algebra D_v. For each positive integer N, we define

$$[N] := \frac{v^N - v^{-N}}{v - v^{-1}}, \ \ [N]^! := [N][N-1]\cdots[2][1].$$

For any integers m, n with $n \geq 0$, we define

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]^!}{[n]^![m-n]^!}.$$

Then it is well-known that $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathcal{A}$ (e.g, see [17, (1.3.1.d)]).

Let $\mathbb{D}_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of D_v generated by (compare with [13, (3.1)])

$$E^{(N)} = \frac{E^N}{[N]^!}, \ F^{(N)} = \frac{F^N}{[N]^!}, \ K^{\pm 1}, \ \widetilde{K}^{\pm 1},$$
$$\begin{bmatrix} K, \widetilde{K} \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K v^{-s+1} - \widetilde{K}^{-1} v^{s-1}}{v^s - v^{-s}},$$

where $N, t \in \mathbb{N} \cup \{0\}$.

For any integers c, t with $t \geq 0$, we define the analogue of $\begin{bmatrix} K, c \\ t \end{bmatrix}$ (see [13, (4.1)]).

$$\begin{bmatrix} K, \widetilde{K}, c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K v^{c-s+1} - \widetilde{K}^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

We have the following (compare it with [14, (4.3.1)]).

Lemma 2.1. *For any non-negative integers a, b, we have that*

$$E^{(a)} F^{(b)} = \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \begin{bmatrix} K, \widetilde{K}, 2t-a-b \\ t \end{bmatrix} E^{(a-t)}.$$

Proof. The equality is proved in a similar way to the proof of [14, (4.3.1)]. We show it by induction on a. For $a = 0$ (or $b = 0$), the claim is trivial. For $a = 1$ and $b > 0$, it also follows from a straightforward verification. Suppose

the formula holds for integers a, b. Then we obtain

$$
\begin{aligned}
&E^{(a+1)}F^{(b)} = \frac{1}{[a+1]}EE^{(a)}F^{(b)}\\
&= \frac{1}{[a+1]}\sum_{0\le t\le \min(a,b)} EF^{(b-t)}\begin{bmatrix} K, \widetilde{K}, 2t-a-b \\ t\end{bmatrix}E^{(a-t)}\\
&= \frac{1}{[a+1]}\Bigg\{\sum_{0\le t\le \min(a,b)} F^{(b-t)}E\begin{bmatrix} K, \widetilde{K}, 2t-a-b \\ t\end{bmatrix}E^{(a-t)}+\\
&\quad \sum_{0\le t\le \min(a,b)} F^{(b-t-1)}\frac{Kv^{1-b+t}-\widetilde{K}^{-1}v^{-1+b-t}}{v-v^{-1}}\begin{bmatrix} K, \widetilde{K}, 2t-a-b \\ t\end{bmatrix}E^{(a-t)}\Bigg\}\\
&= \frac{1}{[a+1]}\Bigg\{\sum_{0\le t\le \min(a,b)} [a-t+1]F^{(b-t)}\begin{bmatrix} K, \widetilde{K}, 2t-a-b-2 \\ t\end{bmatrix}E^{(a-t+1)}+\\
&\quad \sum_{0\le t\le \min(a,b)} F^{(b-t-1)}\frac{Kv^{1-b+t}-\widetilde{K}^{-1}v^{-1+b-t}}{v-v^{-1}}\begin{bmatrix} K, \widetilde{K}, 2t-a-b \\ t\end{bmatrix}E^{(a-t)}\Bigg\}.
\end{aligned}
$$

Thus we have to show that the equality

$$
\begin{aligned}
&\frac{1}{[a+1]}\Bigg\{[a-t+1]F^{(b-t)}\begin{bmatrix} K, \widetilde{K}, 2t-a-b-2 \\ t\end{bmatrix}E^{(a-t+1)}+\\
&F^{(b-t)}\frac{Kv^{-b+t}-\widetilde{K}^{-1}v^{b-t}}{v-v^{-1}}\begin{bmatrix} K, \widetilde{K}, 2t-2-a-b \\ t-1\end{bmatrix}E^{(a-t+1)}\Bigg\}\\
&= F^{(b-t)}\begin{bmatrix} K, \widetilde{K}, 2t-a-1-b \\ t\end{bmatrix}E^{(a+1-t)}
\end{aligned}
$$

holds when $0 \le t \le \min(a, b)$, and that the equality

$$
\begin{aligned}
&\frac{1}{[a+1]}F^{(b-a-1)}\frac{Kv^{-b+a+1}-\widetilde{K}^{-1}v^{b-a-1}}{v-v^{-1}}\begin{bmatrix} K, \widetilde{K}, a-b \\ a\end{bmatrix}\\
&= F^{(b-a-1)}\begin{bmatrix} K, \widetilde{K}, a+1-b \\ a+1\end{bmatrix}
\end{aligned}
$$

holds when $b \ge a + 1$.

The verification of the second equality is straightforward while the first equality follows from the following equation which can be calculated easily:

$$
\begin{aligned}
\begin{bmatrix} K, \widetilde{K}, 2t-a-1-b \\ t\end{bmatrix} = \frac{1}{[a+1]}\Bigg\{&[a-t+1]\begin{bmatrix} K, \widetilde{K}, 2t-a-b-2 \\ t\end{bmatrix}+\\
&\frac{Kv^{-b+t}-\widetilde{K}^{-1}v^{b-t}}{v-v^{-1}}\begin{bmatrix} K, \widetilde{K}, 2t-2-a-b \\ t-1\end{bmatrix}\Bigg\}.
\end{aligned}
$$

□

Let $\mathbb{D}^+_{\mathcal{A}}$ (resp. $\mathbb{D}^-_{\mathcal{A}}$) be the $\mathcal{A}$-subalgebra of $\mathbb{D}_{\mathcal{A}}$ generated by $E^{(a)}$ (resp. $F^{(b)}$), where $a, b \in \mathbb{N} \cup \{0\}$. Let $\mathbb{D}^0_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbb{D}_{\mathcal{A}}$ generated by $K^{\pm 1}$, $\widetilde{K}^{\pm 1}$, $\begin{bmatrix} K, \widetilde{K} \\ t \end{bmatrix}$, $t = 0, 1, 2, \cdots$. We have the following analogues of [15, (2.3),(g8),(g9),(g10))] and [13, (4.1),(d)])[3]

Lemma 2.2. 1) *For any integers c, t, p with $t \geq 0, 0 \leq p \leq t$, we have*

$$v^{-pt} \begin{bmatrix} K, \widetilde{K}, c \\ t \end{bmatrix} = \sum_{j=0}^{p} \begin{bmatrix} K, \widetilde{K}, c-p \\ t-j \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix} \widetilde{K}^{-j} v^{-cj}.$$

In particular, for any integer c, t with $0 \leq c \leq t$, we have

$$\begin{bmatrix} K, \widetilde{K}, c \\ t \end{bmatrix} = \sum_{j=0}^{c} \begin{bmatrix} K, \widetilde{K} \\ t-j \end{bmatrix} \begin{bmatrix} c \\ j \end{bmatrix} \widetilde{K}^{-j} v^{c(t-j)}.$$

2) *For any integers c, t, p with $t \geq 0, p \geq 1$, we have*

$$v^{-pt} \begin{bmatrix} K, \widetilde{K}, -c \\ t \end{bmatrix} = \sum_{j=0}^{t} (-1)^j \begin{bmatrix} K, \widetilde{K}, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj}.$$

In particular, for any integer c, t with $c \geq 1, t \geq 0$, we have

$$\begin{bmatrix} K, \widetilde{K}, -c \\ t \end{bmatrix} = \sum_{j=0}^{t} (-1)^j \begin{bmatrix} K, \widetilde{K} \\ t-j \end{bmatrix} \begin{bmatrix} c+j-1 \\ j \end{bmatrix} K^j v^{c(t-j)}.$$

3) *For any $c \in \mathbb{Z}, t \in \mathbb{N} \cup \{0\}$, $\begin{bmatrix} K, \widetilde{K}, c \\ t \end{bmatrix} \in \mathbb{D}^0_{\mathcal{A}}$.*

4) *For any non-negative integers t, t' with $t \geq 1$, we have that*

$$\begin{bmatrix} t+t' \\ t \end{bmatrix} \begin{bmatrix} K, \widetilde{K} \\ t+t' \end{bmatrix} = \sum_{0 \leq j \leq t'} (-1)^j v^{t(t'-j)} \begin{bmatrix} t+j-1 \\ j \end{bmatrix} K^j \begin{bmatrix} K, \widetilde{K} \\ t \end{bmatrix} \begin{bmatrix} K, \widetilde{K} \\ t'-j \end{bmatrix}.$$

Proof. 1) The second statement follows from induction on t. It suffices to prove the first statement. We show it by induction on p. The case where $p = 0$ is trivial. For $p = 1$, we have the following

$$\begin{aligned}
&\begin{bmatrix} K, \widetilde{K}, c-1 \\ t \end{bmatrix} + \begin{bmatrix} K, \widetilde{K}, c-1 \\ t-1 \end{bmatrix} \widetilde{K}^{-1} v^{-c} \\
&= \left(\prod_{s=1}^{t-1} \frac{K v^{c-1-s+1} - \widetilde{K}^{-1} v^{-c+1+s-1}}{v^s - v^{-s}} \right) \left\{ \frac{K v^{c-t} - \widetilde{K}^{-1} v^{-c+t}}{v^t - v^{-t}} + \widetilde{K}^{-1} v^{-c} \right\} \\
&= v^{-t} \begin{bmatrix} K, \widetilde{K}, c \\ t \end{bmatrix},
\end{aligned}$$

[3]Note that in [15, (2.3),(g10)], the range of j in the summation should be $0 \leq j \leq c$ instead of $0 \leq j \leq t$.

as required. Suppose now the equality holds for $p = N$, we consider the case where $p = N + 1$. We get

$$\begin{aligned}
& v^{-(N+1)t} \begin{bmatrix} K,\, \widetilde{K},\, c \\ t \end{bmatrix} = v^{-Nt} v^{-t} \begin{bmatrix} K,\, \widetilde{K},\, c \\ t \end{bmatrix} \\
&= v^{-Nt} \left\{ \begin{bmatrix} K,\, \widetilde{K},\, c-1 \\ t \end{bmatrix} + \begin{bmatrix} K,\, \widetilde{K},\, c-1 \\ t-1 \end{bmatrix} \widetilde{K}^{-1} v^{-c} \right\} \\
&= v^{-Nt} \begin{bmatrix} K,\, \widetilde{K},\, c-1 \\ t \end{bmatrix} + v^{-N(t-1)} \begin{bmatrix} K,\, \widetilde{K},\, c-1 \\ t-1 \end{bmatrix} \widetilde{K}^{-1} v^{-c-N} \\
&= \sum_{j=0}^{N} \left\{ \begin{bmatrix} K,\, \widetilde{K},\, c-N-1 \\ t-j \end{bmatrix} \begin{bmatrix} N \\ j \end{bmatrix} \widetilde{K}^{-j} v^{-(c-1)j} + \right. \\
&\qquad \left. \begin{bmatrix} K,\, \widetilde{K},\, c-N-1 \\ t-1-j \end{bmatrix} \begin{bmatrix} N \\ j \end{bmatrix} \widetilde{K}^{-j-1} v^{-(c-1)j-c-N} \right\} \\
&= \sum_{j=0}^{N+1} \begin{bmatrix} K,\, \widetilde{K},\, c-N-1 \\ t-j \end{bmatrix} \begin{bmatrix} N+1 \\ j \end{bmatrix} \widetilde{K}^{-j} v^{-cj},
\end{aligned}$$

as desired.

2) Now we use induction on t. The case where $t = 0$ is trivial. For $t = 1$, we obtain the equations

$$\begin{aligned}
& \sum_{j=0}^{1} (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ 1-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj} \\
&= \frac{K v^{p-c} - \widetilde{K}^{-1} v^{c-p}}{v - v^{-1}} - \frac{v^p - v^{-p}}{v - v^{-1}} K v^{-c} \\
&= v^{-p} \frac{K v^{-c} - \widetilde{K}^{-1} v^{c}}{v - v^{-1}} = v^{-p} \begin{bmatrix} K,\, \widetilde{K},\, -c \\ 1 \end{bmatrix},
\end{aligned}$$

as required.

Suppose the equality holds for t, we now consider the equality for $t+1$. We get then

$$\begin{aligned}
& v^{-p(t+1)} \begin{bmatrix} K,\, \widetilde{K},\, -c \\ t+1 \end{bmatrix} = v^{-pt} \begin{bmatrix} K,\, \widetilde{K},\, -c \\ t \end{bmatrix} v^{-p} \frac{K v^{-c-t} - \widetilde{K}^{-1} v^{c+t}}{v^{t+1} - v^{-t-1}} \\
&= \sum_{j=0}^{t} (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj-p} \frac{K v^{-c-t} - \widetilde{K}^{-1} v^{c+t}}{v^{t+1} - v^{-t-1}} \\
&= \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t+1 \end{bmatrix} - \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t \end{bmatrix} K v^{-c-t} \frac{v^p - v^{-p}}{v^{t+1} - v^{-t-1}} + \\
&\quad \sum_{j=1}^{t} (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj-p} \frac{K v^{-c-t} - \widetilde{K}^{-1} v^{c+t}}{v^{t+1} - v^{-t-1}}.
\end{aligned}$$

Note that

$$
\begin{aligned}
&(-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj-p} \frac{Kv^{-c-t} - \widetilde{K}^{-1} v^{c+t}}{v^{t+1} - v^{-t-1}} \\
&= (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj} \Bigg(\frac{Kv^{-c+p-t+j} - \widetilde{K}^{-1} v^{c-p+t-j}}{v^{t+1-j} - v^{j-t-1}} \\
&\quad \frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} - Kv^{-c-t} \frac{v^{p+2j} - v^{-p}}{v^{t+1} - v^{-t-1}} \Bigg) \\
&= (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t+1-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj} \frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} + \\
&\quad (-1)^{j+1} \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^{j+1} v^{-c(j+1)} v^{-t} \frac{v^{p+2j} - v^{-p}}{v^{t+1} - v^{-t-1}}.
\end{aligned}
$$

Now the required equality follows from the following calculation:

$$
\begin{aligned}
&(-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t+1-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj} \frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} + \\
&(-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t-j+1 \end{bmatrix} \begin{bmatrix} p+j-2 \\ j-1 \end{bmatrix} K^j v^{-cj} v^{-t} \frac{v^{p+2j-2} - v^{-p}}{v^{t+1} - v^{-t-1}} \\
&= (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t+1-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj} \Bigg(\frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} + \\
&\quad \frac{[j]}{[p+j-1]} \frac{v^{p+2j-t-2} - v^{-p-t}}{v^{t+1} - v^{-t-1}} \Bigg) \\
&= (-1)^j \begin{bmatrix} K,\, \widetilde{K},\, p-c \\ t+1-j \end{bmatrix} \begin{bmatrix} p+j-1 \\ j \end{bmatrix} K^j v^{-cj}.
\end{aligned}
$$

3) follows from 1) and 2).

4) follows from 2) and the following equality:

$$
\begin{bmatrix} K,\, \widetilde{K} \\ t+t' \end{bmatrix} = \begin{bmatrix} K,\, \widetilde{K} \\ t \end{bmatrix} \begin{bmatrix} K,\, \widetilde{K},\, -t \\ t' \end{bmatrix}.
$$

□

Let θ be the algebra automorphism of D_v which is defined on generators by

$$
\theta(E) = E, \quad \theta(F) = F, \quad \theta(K^{\pm 1}) = \widetilde{K}^{\pm 1}, \quad \theta(\widetilde{K}^{\pm 1}) = K^{\pm 1}.
$$

Since

$$
\theta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ t \end{bmatrix} \Big) = \prod_{s=1}^{t} \frac{\widetilde{K} v^{-s+1} - K^{-1} v^{s-1}}{v^s - v^{-s}} = K^{-t} \widetilde{K}^t \begin{bmatrix} K,\, \widetilde{K} \\ t \end{bmatrix},
$$

it follows that θ restricts to an $\mathcal{A}$-algebra automorphism of $\mathbb{D}_{\mathcal{A}}$. Henceforth, we write

$$\begin{bmatrix} \widetilde{K},\, K \\ t \end{bmatrix} := \theta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ t \end{bmatrix}\Big), \quad \begin{bmatrix} \widetilde{K},\, K,\, c \\ t \end{bmatrix} := \theta\Big(\begin{bmatrix} K,\, \widetilde{K},\, c \\ t \end{bmatrix}\Big).$$

Then one can get a second version of our previous two lemmas by applying the automorphism θ.

Lemma 2.3. *With the notations as above, we have that the $\mathcal{A}$-algebra $\mathbb{D}_{\mathcal{A}}^{+}$ (resp. $\mathbb{D}_{\mathcal{A}}^{-}$) is a free $\mathcal{A}$-module, and the set $\{E^{(a)}\}$ (resp. the set $\{F^{(b)}\}$), where $a, b \in \mathbb{N} \cup \{0\}$, form an $\mathcal{A}$-basis of $\mathbb{D}_{\mathcal{A}}^{+}$ (resp. of $\mathbb{D}_{\mathcal{A}}^{-}$),*

Proof. This follows from the fact that the set $\{E^{(a)}\}$ (resp. the set $\{F^{(a)}\}$) is a basis of $\mathbb{D}_v^{+}$ (resp. of $\mathbb{D}_v^{-}$) and the following equalities:

$$E^{(a)}E^{(b)} = \begin{bmatrix} a+b \\ b \end{bmatrix} E^{(a+b)}, \quad F^{(a)}F^{(b)} = \begin{bmatrix} a+b \\ b \end{bmatrix} F^{(a+b)}.$$

□

We define $\mathbb{D}_{\mathcal{A}}^{\geq 0} := \mathbb{D}_{\mathcal{A}}^{+}\mathbb{D}_{\mathcal{A}}^{0}$, $\mathbb{D}_{\mathcal{A}}^{\leq 0} := \mathbb{D}_{\mathcal{A}}^{-}\mathbb{D}_{\mathcal{A}}^{0}$. According to [8, Remark 3.1], we know that for each positive integer N,

$$\begin{aligned}
\Delta(E^{(N)}) &= \sum_{i=0}^{N} v^{i(N-i)} E^{(N-i)} K^{i} \otimes E^{(i)}, \\
\Delta(F^{(N)}) &= \sum_{i=0}^{N} v^{i(N-i)} F^{(i)} \otimes F^{(N-i)} \widetilde{K}^{-i}, \\
S(E^{(N)}) &= (-1)^{N} v^{(1-N)N} K^{-N} E^{(N)}, \\
S(F^{(N)}) &= (-1)^{N} v^{(N-1)N} F^{(N)} \widetilde{K}^{N}.
\end{aligned}$$

Lemma 2.4. *For any positive integer t, we have*

$$\Delta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ t \end{bmatrix}\Big) = \sum_{a=0}^{t} \begin{bmatrix} K,\, \widetilde{K} \\ t-a \end{bmatrix} \widetilde{K}^{-a} \otimes K^{t-a} \begin{bmatrix} K,\, \widetilde{K} \\ a \end{bmatrix}.$$

Proof. We use induction on t. If $t=1$, we have that

$$
\begin{aligned}
&\Delta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ 1 \end{bmatrix}\Big)\\
&=\Delta\Big(\frac{K-\widetilde{K}^{-1}}{v-v^{-1}}\Big)\\
&=\frac{K\otimes K-\widetilde{K}^{-1}\otimes\widetilde{K}^{-1}}{v-v^{-1}}\\
&=\frac{K-\widetilde{K}^{-1}}{v-v^{-1}}\otimes K+\widetilde{K}^{-1}\otimes\frac{K-\widetilde{K}^{-1}}{v-v^{-1}}\\
&=\begin{bmatrix} K,\, \widetilde{K} \\ t-a \end{bmatrix}\otimes K+\widetilde{K}^{-1}\otimes\begin{bmatrix} K,\, \widetilde{K} \\ t-a \end{bmatrix},
\end{aligned}
$$

as required.

Now assume that the equality holds for $t=N$. We consider the case where $t=N+1$. We have that

$$
\begin{aligned}
&\Delta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ N+1 \end{bmatrix}\Big)\\
&=\Delta\Big(\prod_{s=1}^{N+1}\frac{Kv^{-s+1}-\widetilde{K}^{-1}v^{s-1}}{v^{s}-v^{-s}}\Big)\\
&=\Delta\Big(\begin{bmatrix} K,\, \widetilde{K} \\ N \end{bmatrix}\Big)\Delta\Big(\frac{Kv^{-N}-\widetilde{K}^{-1}v^{N}}{v^{N+1}-v^{-N-1}}\Big)\\
&=\sum_{a=0}^{N}\Big(\begin{bmatrix} K,\, \widetilde{K} \\ N-a \end{bmatrix}\widetilde{K}^{-a}\otimes K^{N-a}\begin{bmatrix} K,\, \widetilde{K} \\ a \end{bmatrix}\Big)\Big(\frac{Kv^{-N}\otimes K-\widetilde{K}^{-1}v^{N}\otimes\widetilde{K}^{-1}}{v^{N+1}-v^{-N-1}}\Big)\\
&=\sum_{a=0}^{N}\Big(\begin{bmatrix} K,\, \widetilde{K} \\ N-a \end{bmatrix}\widetilde{K}^{-a}\otimes K^{N-a}\begin{bmatrix} K,\, \widetilde{K} \\ a \end{bmatrix}\Big)\Big(\frac{Kv^{-N}-\widetilde{K}^{-1}v^{N}}{v^{N+1}-v^{-N-1}}\otimes K+\\
&\qquad\frac{v^{N}\widetilde{K}^{-1}}{[N+1]}\otimes\begin{bmatrix} K,\, \widetilde{K} \\ 1 \end{bmatrix}\Big)\\
&=\sum_{a=0}^{N}\Big(\begin{bmatrix} K,\, \widetilde{K} \\ N-a \end{bmatrix}\widetilde{K}^{-a}\frac{Kv^{-N}-\widetilde{K}^{-1}v^{N}}{v^{N+1}-v^{-N-1}}\otimes K^{N+1-a}\begin{bmatrix} K,\, \widetilde{K} \\ a \end{bmatrix}+\\
&\qquad\frac{v^{N}}{[N+1]}\begin{bmatrix} K,\, \widetilde{K} \\ N-a \end{bmatrix}\widetilde{K}^{-a-1}\otimes K^{N-a}\begin{bmatrix} K,\, \widetilde{K} \\ a \end{bmatrix}\begin{bmatrix} K,\, \widetilde{K} \\ 1 \end{bmatrix}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{a=0}^{N}\Bigg(\begin{bmatrix}K,\ \widetilde{K}\\ N-a\end{bmatrix}\widetilde{K}^{-a}\frac{Kv^{-N}-\widetilde{K}^{-1}v^{N}}{v^{N+1}-v^{-N-1}}\otimes K^{N+1-a}\begin{bmatrix}K,\ \widetilde{K}\\ a\end{bmatrix}+\\
&\qquad \frac{v^{N}}{[N+1]}\begin{bmatrix}K,\ \widetilde{K}\\ N-a\end{bmatrix}\widetilde{K}^{-a-1}\otimes\Big(v^{-a}[a]K^{N+1-a}\begin{bmatrix}K,\ \widetilde{K}\\ a\end{bmatrix}+\\
&\qquad\qquad v^{-a}[a+1]K^{N-a}\begin{bmatrix}K,\ \widetilde{K}\\ a+1\end{bmatrix}\Big)\Bigg)\qquad\text{(by Lemma 2.2)}\\
&= \widetilde{K}^{-N-1}\otimes\begin{bmatrix}K,\ \widetilde{K}\\ N+1\end{bmatrix}+\sum_{a=0}^{N}\Bigg\{\Bigg(\begin{bmatrix}K,\ \widetilde{K}\\ N-a\end{bmatrix}\widetilde{K}^{-a}\frac{Kv^{-N}-\widetilde{K}^{-1}v^{N}}{v^{N+1}-v^{-N-1}}+\\
&\qquad \begin{bmatrix}K,\ \widetilde{K}\\ N-a\end{bmatrix}\frac{v^{N-a}[a]}{[N+1]}\widetilde{K}^{-a-1}+\frac{v^{N-a+1}(v^{a}-v^{-a})}{v^{N+1}-v^{-N-1}}\begin{bmatrix}K,\ \widetilde{K}\\ N-a+1\end{bmatrix}\Bigg)\otimes\\
&\qquad\quad K^{N+1-a}\begin{bmatrix}K,\ \widetilde{K}\\ a\end{bmatrix}\Bigg\}\\
&= \sum_{a=0}^{N+1}\begin{bmatrix}K,\ \widetilde{K}\\ N+1-a\end{bmatrix}\widetilde{K}^{-a}\otimes K^{N+1-a}\begin{bmatrix}K,\ \widetilde{K}\\ a\end{bmatrix}.
\end{aligned}
$$

This proves the lemma. □

Note that a special case of the above result appeared in the proof of [1, Lemma 1.1(ii)]. However, we did not find any specific reference to the above calculations.

Corollary 2.5. *With the notations as above, $\mathbb{D}_{\mathcal{A}}$ is a Hopf $\mathcal{A}$-subalgebra of $\mathbb{D}_v$, and both $\mathbb{D}_{\mathcal{A}}^{\geq 0}$ and $\mathbb{D}_{\mathcal{A}}^{\leq 0}$ are Hopf subalgebras of $\mathbb{D}_{\mathcal{A}}$.*

It is easy to see that $\mathbb{D}_{\mathcal{A}}^{\geq 0}\cong\mathbb{D}_{\mathcal{A}}^{+}\otimes\mathbb{D}_{\mathcal{A}}^{0}$, $\mathbb{D}_{\mathcal{A}}^{\leq 0}\cong\mathbb{D}_{\mathcal{A}}^{-}\otimes\mathbb{D}_{\mathcal{A}}^{0}$. For any field k which is an $\mathcal{A}$-algebra, we define $\mathbb{D}_k := k\otimes_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}$, $\mathbb{D}_k^{\geq 0} := k\otimes_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}^{\geq 0}$, $\mathbb{D}_k^{\leq 0} := k\otimes_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}^{\leq 0}$.

Remark 2.6. *It would be interesting to know if $\mathbb{D}_{\mathcal{A}}^{0}$ is a free $\mathcal{A}$-module and whether there is a triangular decomposition for the $\mathcal{A}$-algebra $\mathbb{D}_{\mathcal{A}}$.*

Corollary 2.7. *We consider $\mathbb{Q}(v)$ as an $\mathcal{A}$-algebra in a natural way. Then the natural map $\mathbb{Q}(v)\otimes_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}\to \mathrm{D}_v$, $a\otimes x\mapsto ax, \forall\, a\in\mathbb{Q}(v), x\in\mathbb{D}_{\mathcal{A}}$, is a $\mathbb{Q}(v)$-algebra isomorphism.*

3. Representations of the algebra D_q

Throughout this section, we assume that k is an algebraically closed field.

Let M be a D_q-module such that $\mathrm{End}_{\mathrm{D}_q}(M) = k$. Note that the element $K\widetilde{K}^{-1}$ is invertible and central in D_q. Therefore, there is an element $0 \neq z \in k$ such that $K\widetilde{K}^{-1}$ acts as the scalar z on M. For each $0 \neq z \in k$, we fix a square root $\sqrt{z}$ of z. Let π_z^+ be the k-algebra homomorphism $\mathrm{D}_q \to \mathrm{U}_q$ which is defined on generators by

$$\pi_z^+(E) = \sqrt{z}E,\ \pi_z^+(F) = F,\ \pi_z^+(K) = \sqrt{z}K,\ \pi_z^+(\widetilde{K}) = \sqrt{z}^{-1}K.$$

It is easy to check that π_z^+ is well-defined. Moreover, the kernel of π_z^+, which is the ideal generated by $K\widetilde{K}^{-1} - z$, annihilates the module M. It follows that M becomes a module over the algebra U_q in a natural way. Similarly, we have a well-defined k-algebra homomorphism $\pi_z^- : \mathrm{D}_q \to \mathrm{U}_q$ which is defined on generators by

$$\pi_z^-(E) = -\sqrt{z}E,\ \pi_z^-(F) = F,\ \pi_z^-(K) = -\sqrt{z}K,\ \pi_z^-(\widetilde{K}) = -\sqrt{z}^{-1}K.$$

Note that π_1^+ is a Hopf algebra map, but in general, both π_z^+ and π_z^- are not Hopf algebra maps.

We call a D_q-module M a weight D_q-module if both K and $\widetilde{K}$ act semisimply on M. In that case, $K\widetilde{K}^{-1}$ acts semisimply on M as well. Similarly, we call a U_q-module N a weight U_q-module if K acts semisimply on N.

Lemma 3.1. *Every finite dimensional simple (resp. indecomposable weight) D_q-module is the pull-back of a finite dimensional simple (resp. indecomposable weight) U_q-module through the algebra homomorphisms $\pi_z^{\pm}$ for some $0 \neq z \in k$.*

Proof. For any finite dimensional D_q-module M, we consider the eigenspace of $K\widetilde{K}^{-1}$ on M. Since $K\widetilde{K}^{-1}$ is central in D_q, we deduce that each such eigenspace must be a D_q-submodule of M. Therefore, if M is a simple D_q-module or an indecomposable weight D_q-module, then $K\widetilde{K}^{-1}$ can have only one eigenvalue on M. This proves that $K\widetilde{K}^{-1}$ acts as a scalar on M, hence the lemma follows immediately from the previous discussion. □

Let M be a U_q-module. For any $0 \neq \lambda \in k$. We denote by M_λ^+ (resp. M_λ^-) the pull-back of M through the algebra homomorphism π_λ^+ (resp. π_λ^-).

Theorem 3.2. *The category $\widetilde{\mathcal{C}}$ of finite dimensional weight D_q-modules is equivalent to a direct sum of $|k^\times|$ copies of the category $\mathcal{C}$ of finite dimensional weight U_q-modules.*

Proof. By definition, every object M of $\mathcal{C}$ is of the form $\oplus_{\lambda\in k^\times} M(\lambda)$, where for each λ, $M(\lambda)$ is a finite dimensional indecomposable weight U_q-module,

and $|\{\lambda\in k^\times \mid M(\lambda)\neq 0\}|<\infty$. We use θ^+ to denote the functor from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ such that

$$\theta^+\Big(\bigoplus_{\lambda\in k^\times} M(\lambda)\Big) := \bigoplus_{\lambda\in k^\times} M(\lambda)^+_\lambda .$$

The action of θ^+ on the set of morphisms is defined in an obvious way. Then applying Lemma 3.1, we see that θ^+ is an equivalence of categories. In a similar way, if we define θ^- to be the functor from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ satisfying

$$\theta^-\Big(\bigoplus_{\lambda\in k^\times} M(\lambda)\Big) := \bigoplus_{\lambda\in k^\times} M(\lambda)^-_\lambda ,$$

and the action of θ^- on the set of morphisms is defined in an obvious way, then θ^- is also an equivalence of categories. □

Let M be a U_q-module. Let $0\neq z\in k$. Let ε_z^+ (resp. ε_z^-) be the one-dimensional representation of D_q which is defined on generators by

$$\varepsilon_z^+(E)=0=\varepsilon_z^+(F),\ \varepsilon_z^+(K)=\sqrt{z},\ \varepsilon_z^+(\widetilde{K})=\sqrt{z}^{-1}.$$

$\Big($ resp. $\varepsilon_z^-(E)=0=\varepsilon_z^-(F),\ \varepsilon_z^-(K)=-\sqrt{z},\ \varepsilon_z^-(\widetilde{K})=-\sqrt{z}^{-1}.\Big)$ It is easy to check that both ε_z^+ and ε_z^- are well-defined. For any $z,z'\in k^\times$, we have that

$$\varepsilon_z^\pm\otimes\varepsilon_{z'}^\pm\cong\varepsilon_{z'}^\pm\otimes\varepsilon_z^\pm\cong\begin{cases}\varepsilon_{zz'}^+, & \text{if } \sqrt{z}\sqrt{z'}=\sqrt{zz'};\\ \varepsilon_{zz'}^-, & \text{if } \sqrt{z}\sqrt{z'}=-\sqrt{zz'},\end{cases}$$

$$\varepsilon_z^+\otimes\varepsilon_{z'}^-\cong\varepsilon_{z'}^-\otimes\varepsilon_z^+\cong\begin{cases}\varepsilon_{zz'}^+, & \text{if } \sqrt{z}\sqrt{z'}=-\sqrt{zz'};\\ \varepsilon_{zz'}^-, & \text{if } \sqrt{z}\sqrt{z'}=\sqrt{zz'}.\end{cases}$$

Lemma 3.3. *Let $0\neq z\in k$, let M be a U_q-module. Then*

1) $M_z^+\cong\varepsilon_z^+\otimes M_1$;

2) $\varepsilon_z^+\otimes M_1\cong M_1\otimes\varepsilon_z^+$ if and only if $\varepsilon_z^-\otimes M_1\cong M_1\otimes\varepsilon_z^-$, in that case, for any $0\neq z'\in k$ and any U_q-module N, we have that

$$M_z^\pm\otimes N_{z'}^\pm\cong\begin{cases}(M\otimes N)_{zz'}^+, & \text{if } \sqrt{z}\sqrt{z'}=\sqrt{zz'};\\ (M\otimes N)_{zz'}^-, & \text{if } \sqrt{z}\sqrt{z'}=-\sqrt{zz'};\end{cases}$$

$$M_z^\pm\otimes N_{z'}^\mp\cong\begin{cases}(M\otimes N)_{zz'}^+, & \text{if } \sqrt{z}\sqrt{z'}=-\sqrt{zz'};\\ (M\otimes N)_{zz'}^-, & \text{if } \sqrt{z}\sqrt{z'}=\sqrt{zz'};\end{cases}$$

3) $\varepsilon_z^\pm\otimes M_1\cong M_1\otimes\varepsilon_z^\pm$ if and only if $\varepsilon_z^\pm\otimes\theta^+(M)\cong\theta^+(M)\otimes\varepsilon_z^\pm$. The same is true if we replace "θ^+" by "θ^-".

Proof. The first statement follows from direct verification. The second and the third statements follow from the associativity of the tensor product and the previous discussion. □

Note that in general, the assumption $\varepsilon_z^+ \otimes M_1 \cong M_1 \otimes \varepsilon_z^+$ in Lemma 3.3 (2) may not hold. However, it does hold in the following two cases:

Case 1. q is not a root of unity, M is an integrable weight module over $\mathrm{U_q}$, i.e., both E, F act locally nilpotently on M and both K, K^{-1} act semisimply on M. In this case, we claim that the D_q-modules $\varepsilon_z^+ \otimes M_1$ and $M_1 \otimes \varepsilon_z^+$ are isomorphic to each other. In fact, since every integrable weight module over $\mathrm{U_q}$ is completely reducible, we can reduce the proof to the case where M is an irreducible highest weight module over $\mathrm{U_q}$. Then $\varepsilon_z^+ \otimes M_1 \cong M_z^+$ is an irreducible D_q-module. Note that the central element $K\widetilde{K}^{-1}$ acts as the same scalar on both $\varepsilon_z^+ \otimes M_1$ and $M_1 \otimes \varepsilon_z^+$. It follows that the D_q-action on both of these two modules can factored through the surjective homomorphism $\pi_{z'}^+$ for some $z' \in k^\times$. So these two modules can be naturally regarded as integrable weight modules over $\mathrm{U_q}$. On the other hand, it is well-known that the isomorphism class of an integrable weight module over $\mathrm{U_q}$ is completely determined by its characters. Since both $M_1 \otimes \varepsilon_z^+$ and $\varepsilon_z^+ \otimes M_1$ have the same characters, they must be isomorphic to each other as $\mathrm{U_q}$-modules, and hence are also isomorphic to each other as D_q-modules.

Case 2. q is a primitive dth root of unity, $(\sqrt{z})^d = 1$, M is a weight D_q-module such that all of the elements E^d, F^d, $K^d - 1$ act as 0 on M. In this case, both ε_z^+ and M_1 can be regarded as modules over the quotient algebra

$$D_q/\langle E^d, F^d, K^d - 1, \widetilde{K}^d - 1\rangle.$$

Note that the algebra $D_q/\langle E^d, F^d, K^d - 1, \widetilde{K}^d - 1\rangle$ is actually a Hopf algebra, and the natural homomorphism from D_q onto $D_q/\langle E^d, F^d, K^d - 1, \widetilde{K}^d - 1\rangle$ is indeed a Hopf algebra homomorphism, and $D_q/\langle E^d, F^d, K^d - 1, \widetilde{K}^d - 1\rangle$ is indeed isomorphic to the Drinfel'd double of a Taft algebra. Hence it is a quasi-triangular Hopf algebra. As a consequence, both $M_1 \otimes \varepsilon_z^+$ and $\varepsilon_z^+ \otimes M_1$ are isomorphic to each other as modules over $D_q/\langle E^d, F^d, K^d - 1, \widetilde{K}^d - 1\rangle$, and hence are also isomorphic to each other as D_q-modules.

We use $\widetilde{\mathcal{C}}_0$ to denote the full subcategory of all the finite dimensional weight D_q-modules $\widetilde{M}$ satisfying $\varepsilon_z^+ \otimes \widetilde{M} \cong \widetilde{M} \otimes \varepsilon_z^+$ for any $z \in k^\times$, and we use $\mathcal{C}_0$ to denote the full subcategory of all the finite dimensional weight $\mathrm{U_q}$-modules M satisfying $\varepsilon_z^+ \otimes M_1 \cong M_1 \otimes \varepsilon_z^+$ for any $z \in k^\times$.

Lemma 3.3 (2) provides a very easy solution to the problem of decomposing the tensor product of certain D_q-modules, i.e., reducing them to the corresponding problem for U_q-modules, where it has been extensively studied and the results are well-known, see [19], [23] and [26]. Therefore, a large part of the representations (including all irreducible representations) of the quantum double D_q can be realized as certain pullback from the representations of the quantized enveloping algebra U_q. Note that the representations of the

quantized enveloping algebra U_q is well-understood (cf. [8]). In particular, the tensor product of finite dimensional simple D_q-modules is determined.

In the following, we shall summarize some results and corollaries for the algebra D_q. We mainly follow the formulation given in [8]. We fix a $0 \neq z \in k$. For each $0 \neq \lambda \in k$, let $M(\lambda)$ be the U_q-module defined in [8, (2.4)]. By pulling back through $\pi_z^{\pm}$, we get a D_q-module $M_z^{\pm}(\lambda)$. We call it the Verma modules associated to (z, λ). We have the following two results concerning Verma modules and simple modules over D_q (compare with [8, (2.4), (2.5)]).

Corollary 3.4. *With the notations as above,*

$$M_z^{+}(\lambda) \cong \mathrm{D}_q/(\mathrm{D}_q E + \mathrm{D}_q(K - \sqrt{z}\lambda) + \mathrm{D}_q(\widetilde{K} - \sqrt{z}^{-1}\lambda)),$$

and there is a k-basis $\{m_i\}_{i=o}^{\infty}$ of $M_z(\lambda)$ such that for all i,

$$\begin{aligned}
&Km_i = \sqrt{z}\lambda q^{-2i} m_i, \ \widetilde{K} m_i = \sqrt{z}^{-1}\lambda q^{-2i} m_i,\\
&Fm_i = m_{i+1},\\
&Em_i = \begin{cases} 0, & \text{if } i = 0,\\ [i]_q \sqrt{z}\dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1}, & \text{otherwise,} \end{cases}
\end{aligned}$$

where

$$[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}.$$

The result for $M_z^{-}(\lambda)$ is similar.

Corollary 3.5. *Suppose that q is not a root of unity in k and $0 \neq \lambda \in k$. If $\lambda \neq \pm q^n$ for all integers $n \geq 0$, then the D_q-module $M_z^{\pm}(\lambda)$ is simple. If $\lambda = \pm q^n$ for some integers $n \geq 0$, then $M_z^{\pm}(\lambda)$ has a unique maximal proper submodule which is spanned by all m_i with $i \geq n+1$ and is isomorphic to $M_z^{\pm}(q^{-2(n+1)}\lambda)$. In this case, the quotient of $M_z^{\pm}(\lambda)$ modulo the maximal proper submodule is an $(n+1)$-dimensional simple D_q-module.*

Suppose that q is not a root of unity in k. By Corollary 3.5, we know that if $\lambda = q^n$ for some integers $n \geq 0$, we get two $(n+1)$-dimensional simple D_q-modules, we denote it by $L_z^{+}(n,+), L_z^{-}(n,+)$; while if $\lambda = -q^n$ for some integers $n \geq 0$, we get two $(n+1)$-dimensional simple D_q-modules, denoted by $L_z^{+}(n,-), L_z^{-}(n,-)$. Note that $L_z^{+}(n,+) \cong L_z^{-}(n,-)$, $L_z^{-}(n,+) \cong L_z^{+}(n,-)$. Therefore, we define

$$L_z(n,+) := L_z^{+}(n,+), \ \ L_z(n,-) := L_z^{+}(n,-).$$

In fact, the simple D_q-module $L_z(n,+)$ (resp. $L_z(n,-)$) is just the pull-back of simple U_q-module $L(n,+)$ (resp. $L(n,-)$) through the k-algebra homomorphism π_z^{+}, see [8, Theorem 2.6] for the definitions of $L(n,+)$ and $L(n,-)$.

By construction, $L_z(n,+)$ has a basis $\{m_i\}_{i=0}^n$ such that

$$\begin{aligned}
&Km_i = z^{1/2}q^{n-2i}m_i,\ \widetilde{K}m_i = z^{-1/2}q^{n-2i}m_i,\\
&Fm_i = m_{i+1},\\
&Em_i = \begin{cases} 0, & \text{if } i=0,\\ z^{1/2}[i]_q[n+1-i]_q m_{i-1}, & \text{otherwise.}\end{cases}
\end{aligned}$$

Similarly, $L_z(n,-)$ has a basis $\{m_i'\}_{i=0}^n$ such that

$$\begin{aligned}
&Km_i' = -z^{1/2}q^{n-2i}m_i',\ \widetilde{K}m_i' = -z^{-1/2}q^{n-2i}m_i',\\
&Fm_i' = m_{i+1}',\\
&Em_i' = \begin{cases} 0, & \text{if } i=0,\\ -z^{1/2}[i]_q[n+1-i]_q m_{i-1}', & \text{otherwise.}\end{cases}
\end{aligned}$$

Note that $L_z(n,+) \ncong L_z(n,-)$. In fact, $L_z(n,+) \cong \varepsilon_1^- \otimes L_z(n,-)$, where ε_1^- is the one-dimensional representation of D_q which is defined on generators by $\varepsilon_1^-(E) = 0 = \varepsilon_1^-(F),\ \varepsilon_1^-(K) = -1 = \varepsilon_1^-(\widetilde{K})$.

Corollary 3.6. *Suppose that q is not a root of unity in k. If $\operatorname{ch} k \neq 2$, then the set*

$$\Big\{L_z(n,+), L_z(n,-)\ \Big|\ 0 \neq z \in k, n \in \mathbb{N}\cup\{0\}\Big\}.$$

is a complete set of pairwise inequivalent finite-dimensional simple weight D_q-modules; while if $\operatorname{ch} k = 2$, then the set

$$\Big\{L_z(n,+) = L_z(n,-)\ \Big|\ 0 \neq z \in k, n \in \mathbb{N}\cup\{0\}\Big\}.$$

is a complete set of pairwise inequivalent finite-dimensional simple weight D_q-modules.

Proof. This follows from Lemma 3.1 and [8, Theorem 2.6]. □

Radford in [20] constructed a class of simple Yetter–Drinfel'd (shortly YD) modules for a graded Hopf algebra $H = \oplus_{n=0}^{\infty} H_n$ with H_0 both commutative and cocommutative. When H is finitely graded over an algebraically closed field and H_0 is the group algebra of a finite abelian group, then all simple YD H-modules are in Radford's class of simple YD modules. The Borel subalgebra $\mathrm{U}_q^{\leq 0}$, denoted H_ω ($\omega = q^{-2}, a = K^{-1}$) in [20], is a simple pointed graded Hopf algebra, but not finitely graded. Thus we don't know whether Radford's class of simple YD modules of $\mathrm{U}_q^{\leq 0}$ [20, Proposition 4,(b)], parameterized by $k^\times \times \mathbb{Z}$, contains all YD simple $\mathrm{U}_q^{\leq 0}$-modules. However, we know that Radford's class forms a proper subset of simple D_q-modules. Recall the Hopf algebra map θ' we introduced in Lemma 1.1. Using θ' and noting that the multiplication rule for our quantum double is compatible with the multiplication rule given in [12, Chapter IX, (4.3)] for the Drinfel'd quantum double of finite dimensional Hopf algebras, one sees easily that every

YD $\mathrm{U}_q^{\leq 0}$-module naturally becomes a D_q-module. To keep in accordance with the notations used in [20, Proposition 4], we set $a = \widetilde{K}^{-1}, x = F$, $g = a^l, \omega = q^{-2}$, and let $\beta : \mathrm{U}_q^{\leq 0} \to K$ be an algebra homomorphism.

Proposition 3.7. *With the notations as above and suppose that q is not a root of unity in k, then*

1) if $\beta(a) \neq \omega^{l+n}$ for any integer $n \geq 0$, then $\lambda^2 \neq q^{2n}$ for any integer $n \geq 0$, where $\lambda := \sqrt{\beta(a)}^{-1} q^{-l}, z := \beta(a) q^{-2l}$, in this case, the module $H_{\beta,kg}$ defined in [20, Corollary 1] *is isomorphic (as D_q-module) to the infinite dimensional simple D_q-module $M_z^+(\lambda)$;*

2) if $\beta(a) = \omega^{l+n}$ for some integer $n \geq 0$, we set $z = q^{-2(n+2l)}$, then the module $H_{\beta,kg}$ defined in [20, Corollary 1] *is isomorphic (as D_q-module) to the $(n+1)$-dimensional simple D_q-module $L_z(n,+)$ if $q^{-n-2l} = \sqrt{z}$; or to the $(n+1)$-dimensional simple D_q-module $L_z(n,-)$ if $q^{-n-2l} = -\sqrt{z}$.*

Proof. 1) Let $\lambda := \sqrt{\beta(a)}^{-1} q^{-l}, z := \beta(a) q^{-2l}$. Then it is obvious that $\beta(a) \neq \omega^{l+n}$ for any integer $n \geq 0$ if and only if $\lambda^2 \neq q^{2n}$ for any integer $n \geq 0$. In this case, we know that (by Corollary 3.5) $M_z^+(\lambda)$ is a simple D_q-module. By [20, Proposition 4, (a)] and the formula given in the paragraph below [20, Proposition 4], we have that

$$\begin{cases} \widetilde{K}^{-1} \bullet_\beta g = a \bullet_\beta g = \beta(a) g, \ K \bullet_\beta g = \varphi(K, K^{-l}) g = q^{-2l} g, \\ \quad E \bullet_\beta g = \varphi(E, K^{-l}) g = 0. \end{cases}$$

On the other hand, by the formula given in the paragraph above Corollary 3.5, we have that

$$\begin{cases} \widetilde{K}^{-1} m_0 = z\lambda^{-1} m_0 = \beta(a) m_0, \ K m_0 = \lambda m_0 = q^{-2l} m_0, \\ \quad E m_0 = 0. \end{cases}$$

By the universal property of the D_q-module $M_z^+(\lambda)$ (see Corollary 3.5), we deduce that the map which sends m_0 to g can be uniquely extended to a homomorphism η from $M_z^+(\lambda)$ to $H_{\beta,kg}$. Comparing the action of F on the basis $\{x^i \bullet_\beta g\}_{i=0}^{\infty}$ given in the paragraph below [20, Proposition 4] and the action of F on the basis $\{m_i\}_{i=0}^{\infty}$ given in the paragraph above Corollary 3.5, we know that $\eta(m_i) = x^i \bullet_\beta g$ for each $i \geq 0$, hence η is an isomorphism, as required.

2) We consider only the case where $q^{-n-2l} = \sqrt{z}$, the other case is similar.

By the formula given in the paragraph below [20, Proposition 4], we have that

$$\begin{cases} \widetilde{K}^{-1} \bullet_\beta g = a \bullet_\beta g = \omega^{l+n} g = q^{-2(l+n)} g, \\ \quad K \bullet_\beta g = \varphi(K, K^{-l}) g = q^{-2l} g, \\ \quad E \bullet_\beta g = \varphi(E, K^{-l}) g = 0. \end{cases}$$

On the other hand, by the formula given in the second paragraph below Corollary 3.5, we have that

$$\widetilde{K}^{-1}m_0 = q^{-2(l+n)},\ Km_0 = q^{-2l}m_0,\ Em_0 = 0.$$

By the universal property of the D_q-modules $M_z^+(q^{-2l}), L_z(n,+)$ (see Corollary 3.5), we deduce that the map which sends m_0 to g can be uniquely extended to a homomorphism η' from $M_z^+(q^{-2l})$ to $H_{\beta,kg}$ and hence gives rises to a homomorphism η' from $L_z(n,+)$ to $H_{\beta,kg}$. Comparing the action of F on the basis $\{x^i \bullet_\beta g\}_{i=0}^n$ given in the paragraph below [20, Proposition 4] and the action of F on the basis $\{m_i\}_{i=0}^n$ given in the second paragraph below Corollary 3.5, we know that $\eta'(m_i) = x^i \bullet_\beta g$ for each $i \geq 0$, hence η' is an isomorphism, as required. □

It would be interesting to know if every simple D_q-module comes from a simple YD $\mathrm{U}_q^{\leq 0}$-module when q is not a root of unity. If this is the case, we would know all simple YD $\mathrm{U}_q^{\leq 0}$-modules. Taking the advantage of the well established representation theory of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, we can easily obtain the decomposition of the tensor product of two finite dimensional D_q-modules while it might be difficult for the YD module setting of Radford in [20].

Theorem 3.8. *Suppose that q is not a root of unity in k. Let $z, z' \in k^\times$. Let $m, n \in \mathbb{N} \cup \{0\}$. Then there is a decomposition of D_q-modules:*

$$L_z(m,\pm) \otimes L_{z'}(n,\pm) \cong \begin{cases} \oplus_{i=0}^{\min(m,n)} L_{zz'}(m+n-2i,+) & \text{if } \sqrt{z}\sqrt{z'} = \sqrt{zz'} \\ \oplus_{i=0}^{\min(m,n)} L_{zz'}(m+n-2i,-) & \text{if } \sqrt{z}\sqrt{z'} = -\sqrt{zz'} \end{cases}$$

$$L_z(m,\pm) \otimes L_{z'}(n,\mp) \cong \begin{cases} \oplus_{i=0}^{\min(m,n)} L_{zz'}(m+n-2i,-) & \text{if } \sqrt{z}\sqrt{z'} = \sqrt{zz'} \\ \oplus_{i=0}^{\min(m,n)} L_{zz'}(m+n-2i,+) & \text{if } \sqrt{z}\sqrt{z'} = -\sqrt{zz'} \end{cases}$$

Proof. The theorem follows easily from Lemma 3.3 and the Clebsch–Gordan formula for $L(n,\pm) \otimes L(m,\pm)$. □

Many results for the quantized enveloping algebra $\mathrm{U_q}$ have their analogues for the algebra D_q. For example, it is not hard to show that the element $C = FE + \dfrac{Kq + \widetilde{K}^{-1}q^{-1}}{(q-q^{-1})^2}$ is equal to $EF + \dfrac{Kq^{-1} + \widetilde{K}^{-1}q}{(q-q^{-1})^2}$, and C is in the center of D_q (compare with [8, (2.7)]). In [8, §2.13], the classification of the finite dimensional simple $\mathrm{U_q}$-modules are given when q is a primitive l-th root of unity with l odd (the even case is also similar). As a consequence, we have an analogous classification result for the algebra D_q. For example, when q^2 is a primitive d-th root of unity in k with $d > 1$, we still have the well-defined simple D_q-modules $L_z(n,\pm)$ for $0 \neq z \in k$ and any integer n with $0 \leq n < d$.

Lemma 3.9. *Let q^2 be a primitive d-th root of unity in k with $d > 1$. If M is a finite dimensional simple weight D_q-module such that both E^d and F^d act as 0 on M, then M is isomorphic to one of the following modules:*

$$Z^{\pm}_{0,z}(\lambda),\ L_z(n,+),\ L_z(n,-),\ 0 \neq z \in k,\ 0 \leq n < d,$$

where $0 \neq z \in k, 0 \neq \lambda \in k$ with $\lambda^{2d} \neq 1$, and

$$Z^{\pm}_{0,z}(\lambda) := M^{\pm}_z(\lambda)/(\mathrm{D}_q m_d).$$

4. Connections with the Drinfel'd double of the Taft algebra

Throughout this section, we assume that k is an algebraically closed field, $1 < d \in \mathbb{N}$ and that $q^2 \in k$ is a primitive d-th root of unity.

We consider the quantized enveloping algebra U_q. It is well-known that the elements E^d, F^d, K^d are central in the algebra U_q. Let $\overline{\mathrm{U}}_q^{\geq 0}$ (resp. $\overline{\mathrm{U}}_q^{\leq 0}$) be the quotient of the algebra $\mathrm{U}_q^{\geq 0}$ (resp. $\mathrm{U}_q^{\leq 0}$) modulo the ideal generated by $E^d, K^d - 1$ (resp. by $F^d, K^d - 1$). It is well-known that the ideal generated by $E^d, K^d - 1$ (resp. by $F^d, K^d - 1$) is a Hopf ideal. Hence the algebra $\overline{\mathrm{U}}_q^{\geq 0}$ (resp. the algebra $\overline{\mathrm{U}}_q^{\leq 0}$) is a quotient Hopf algebra of $\mathrm{U}_q^{\geq 0}$ (resp. of $\mathrm{U}_q^{\leq 0}$). Recall the skew Hopf pairing between $\mathrm{U}_q^{\geq 0}$ and $\mathrm{U}_q^{\leq 0}$ defined in Section 1.

Lemma 4.1. *The elements $E^d, K^d - 1 \in \mathrm{U}_q^{\geq 0}, F^d, K^d - 1 \in \mathrm{U}_q^{\leq 0}$ lie in the radical of the skew Hopf pairing. Moreover, the induced skew Hopf pairing between $\overline{\mathrm{U}}_q^{\geq 0}$ and $\overline{\mathrm{U}}_q^{\leq 0}$ is non-degenerate.*

Proof. For convenience, we still denote by E^aK^b the canonical image of $E^aK^b \in \mathrm{U}_q^{\geq 0}$ in $\overline{\mathrm{U}}_q^{\geq 0}$, and do the same for the elements $F^aK^b \in \mathrm{U}_q^{\leq 0}$. Note that the monomials $\{E^aK^b\}_{0 \leq a,b < d}$ (resp. $\{F^aK^b\}_{0 \leq a,b < d}$) form a k-basis of $\overline{\mathrm{U}}_q^{\geq 0}$ (resp. $\overline{\mathrm{U}}_q^{\leq 0}$). Recall that for the skew Hopf pairing φ between $\mathrm{U}_q^{\geq 0}$ and $\mathrm{U}_q^{\geq 0}$,

$$\varphi(E^aK^b, F^{a'}K^{b'}) = 0 \text{ unless } a = a' \text{ (cf. [17, Proposition 1.2.3(d)])}$$

With this in mind, the first statement of the lemma follows from a direct verification. It remains to show that the induced skew Hopf pairing is non-degenerate.

Suppose that $x := \sum_{0 \leq a,b < d} \lambda_{a,b} E^aK^b \in \overline{\mathrm{U}}_q^{\geq 0}$ (where $\lambda_{a,b} \in k$ for each a, b) lies in the radical of the induced skew Hopf pairing between $\overline{\mathrm{U}}_q^{\geq 0}$ and $\overline{\mathrm{U}}_q^{\leq 0}$. We want to show that $\lambda_{a,b} = 0$ for all a, b.

Let $0 \leq a < d$ be a fixed integer. By assumption, we have that

$$0 = \varphi(x, F^a K^{b'}) = \sum_{0 \leq b < d} \lambda_{a,b} \varphi(E^a K^b, F^a K^{b'}), \text{ for } b' = 0, 1, 2, \cdots, d-1.$$

It is not hard to calculate that

$$\varphi(E^a K^b, F^a K^{b'}) = q^{-2bb'} [a]_q^! \Big(\frac{1}{1-q^2}\Big)^a.$$

Since q^2 is a primitive dth root of unity, it follows that $[a]_q^! \neq 0$. Hence we get that

$$\sum_{0 \leq b < d} \lambda_{a,b} q^{-2bb'} = 0, \quad \text{for } b' = 0, 1, 2, \cdots, d-1.$$

Note that the coefficient matrix of the above system of linear equations is the Vandermonde matrix:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & q^{-2} & q^{-4} & \cdots & q^{-2(d-1)} \\ 1 & (q^{-2})^2 & (q^{-4})^2 & \cdots & (q^{-2(d-1)})^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (q^{-2})^{d-1} & (q^{-4})^{d-1} & \cdots & (q^{-2(d-1)})^{d-1} \end{pmatrix},$$

which has the non-zero determinant. It follows that $\lambda_{a,b} = 0$ for all $0 \leq a, b < d$ and hence $x = 0$ as desired. In a similar way, one can prove that if $y := \sum_{0 \leq a,b < d} \lambda_{a,b} F^a K^b \in \overline{\mathrm{U}}_q^{\leq 0}$ (where $\lambda_{a,b} \in k$ for each a, b) lies in the radical of the induced skew Hopf pairing between $\overline{\mathrm{U}}_q^{\geq 0}$ and $\overline{\mathrm{U}}_q^{\leq 0}$, then $y = 0$. This completes the proof of the lemma. $\square$

Since $\overline{\mathrm{U}}_q^{\leq 0}$ is of finite dimension, we have the following consequence of Lemma 4.1.

Corollary 4.2. *With the above induced skew Hopf pairing, the associated quantum double of* $\overline{\mathrm{U}}_q^{\geq 0}$ *and* $\overline{\mathrm{U}}_q^{\leq 0}$ *is isomorphic to the usual Drinfel'd double* (cf. [7], [12]) *of* $\overline{\mathrm{U}}_q^{\leq 0}$ *as a finite-dimensional k-Hopf algebra.*

Denote by $\overline{\mathrm{D}}_q$ the quantum double of $\overline{\mathrm{U}}_q^{\geq 0}$ and $\overline{\mathrm{U}}_q^{\leq 0}$ under the above skew Hopf pairing. Note that the ideal generated by $E^d,\ F^d,\ K^d - 1,\ \widetilde{K}^d - 1$ is a Hopf ideal of D_q.

Theorem 4.3. *As a Hopf algebra,* $\overline{\mathrm{D}}_q$ *is isomorphic to the quotient of* D_q *modulo the ideal generated by* $E^d,\ F^d,\ K^d - 1,\ \widetilde{K}^d - 1$.

Note that $\overline{\mathrm{U}}_q^{\geq 0}$ and $\overline{\mathrm{U}}_q^{\leq 0}$ are isomorphic as k-Hopf algebras. Thus $\overline{\mathrm{U}}_q^{\leq 0}$ is a self-dual Hopf algebra. This Hopf algebra is usually called the Taft algebra, denoted by $T_d(q^{-2})$, as it was constructed in [24] as an interesting class of pointed Hopf algebras.

In [2], Chen classified the irreducible representations of the Drinfel'd double of $T_d(q^{-2})$ and studied their tensor products. We remark that most of the results obtained in [2] can be recovered easily from our Lemma 3.3 and the discussion below Lemma 3.3, and the corresponding known results for $\mathrm{U}_\mathrm{q}(\mathfrak{sl}_2)$. For example, our Lemma 3.9 recovers the classification of simple $D(T_d(q^{-2}))$-modules obtained in [2, Proposition 2.4, Theorem 2.5]. The decomposition formula for the tensor product of two simple $D(T_d(q^{-2}))$-modules obtained in [2, Theorem 3.1] follows easily from our Lemma 3.3 and [23, Theorem 4.5] (see also [19]). Moreover, some results about finite dimensional indecomposable representations of $D(T_d(q^{-2}))$ in [3] can be recovered from Theorem 4.3 and the results from [26].

5. Generalization to the case of arbitrary Cartan matrix

Our main results in Section 3 allow a direct generalization to the case of arbitrary Cartan matrix. To be precise, let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a $n\times n$ matrix with entries in $\{-3,-2,-1,0,2\}$, $a_{i,i} = 2$ and $a_{i,j} \leq 0$ for $i \neq j$. Suppose $(d_1,\dots,d_n)$ is a vector with entries $d_i \in \{1,2,3\}$ such that the matrix $(d_i a_{i,j})$ is symmetric and positive definite. Then A is a Cartan matrix. Let $\alpha_1,\dots,\alpha_n$ be the set of simple roots in the corresponding root system.

Let k be a field. Let q be an invertible element in k satisfying $q^{2d_i} \neq 1$ for every $1 \leq i \leq n$. The quantized enveloping algebra U_q associated to the Cartan matrix $A = (a_{i,j})_{1\leq i,j\leq n}$ (cf. [6], [9] and [10]) is the associative k-algebra with generators $E_i, F_i, K_i, K_i^{-1} (1 \leq i \leq n)$ and the relations:

$$
\begin{aligned}
&K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1 = K_i^{-1}K_i,\\
&K_iE_j = q^{d_ia_{i,j}}E_jK_i,\ K_iF_j = q^{-d_ia_{i,j}}F_jK_i,\\
&E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},\\
&\sum_{r+s=1-a_{i,j}} (-1)^s \begin{bmatrix} 1-a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} E_i^rE_jE_i^s = 0, \quad \text{if } i \neq j,\\
&\sum_{r+s=1-a_{i,j}} (-1)^s \begin{bmatrix} 1-a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} F_i^rF_jF_i^s = 0, \quad \text{if } i \neq j.
\end{aligned}
$$

U_q is a Hopf algebra with comultiplication, counit and antipode given by:

$$
\begin{aligned}
&\Delta(E_i) = E_i\otimes 1 + K_i\otimes E_i,\ \Delta(F_i) = F_i\otimes K_i^{-1} + 1\otimes F_i,\ \Delta(K_i) = K_i\otimes K_i,\\
&\varepsilon(E_i) = 0 = \varepsilon(F_i),\ \varepsilon(K_i) = 1 = \varepsilon(K_i^{-1}),\\
&S(E_i) = -K_i^{-1}E_i,\ S(F_i) = -F_iK_i,\ S(K_i) = K_i^{-1}.
\end{aligned}
$$

Let U_q^+ (resp. U_q^-) be the k-subalgebra of U_q generated by $E_i, 1 \leq i \leq n$ (resp. by $F_i, 1 \leq i \leq n$). Let U_q^0 be the k-subalgebra of U_q generated by $K_i, K_i^{-1}, 1 \leq i \leq n$. Let $\mathrm{U}_q^{\geq 0} := \mathrm{U}_q^+\mathrm{U}_q^0$, $\mathrm{U}_q^{\leq 0} := \mathrm{U}_q^-\mathrm{U}_q^0$. Then both $\mathrm{U}_q^{\geq 0}$ and $\mathrm{U}_q^{\leq 0}$ are Hopf k-subalgebras of U_q. For any monomials

$$(\prod_{1\leq i\leq n} E_i^{a_i})(\prod_{1\leq i\leq n} K_i^{b_i}) \in \mathrm{U}_q^+, \quad (\prod_{1\leq i\leq n} F_i^{a_i})(\prod_{1\leq i\leq n} K_i^{b_i}) \in \mathrm{U}_q^-,$$

we endow them the weights $\sum_{i=1}^n a_i\alpha_i, -\sum_{i=1}^n a_i\alpha_i$ respectively. Like the Hopf pair $(U_q(\mathfrak{sl}_2)^{\geq 0}), U_q(\mathfrak{sl}_2)^{\leq 0})$, there exists a unique pairing $\varphi : \mathrm{U}_q^{\geq 0} \times \mathrm{U}_q^{\leq 0} \to k$ (see [17], [11] and [25]) such that

(1) $\varphi(1, 1) = 1,\ \varphi(1, K_i) = 1 = \varphi(K_i, 1)$, for $1 \leq i \leq n$,

(2) $\varphi(x, y) = 0$, if x, y are homogeneous with different weights,

(3) $\varphi(E_i, F_j) = \delta_{i,j}\dfrac{1}{q^{2d_i} - 1}$, for $1 \leq i, j \leq n$,

(4) $\varphi(K_i, K_j) = q^{d_ia_{i,j}},\ \varphi(K_i, K_j^{-1}) = q^{-d_ia_{i,j}}$, for $1 \leq i, j \leq n$,

(5) $\varphi(x, y'y'') = \varphi(\Delta^{\mathrm{op}}(x), y' \otimes y''),$ for all $x \in \mathrm{U}_q^{\geq 0}$, $y', y'' \in \mathrm{U}_q^{\leq 0}$,

(6) $\varphi(xx', y'') = \varphi(x \otimes x', \Delta(y'')),$ for all $x, x' \in \mathrm{U}_q^{\geq 0}$, $y'' \in \mathrm{U}_q^{\leq 0}$,

(7) $\varphi(S(x), y) = \varphi(x, S^{-1}(y)),$ for all $x \in \mathrm{U}_q^{\geq 0}$, $y \in \mathrm{U}_q^{\leq 0}$.

In other words, $(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0}, \varphi)$ forms a skew Hopf pairing. Thus (as in Section 1) we can make $D(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0}) := \mathrm{U}_q^{\geq 0} \otimes \mathrm{U}_q^{\leq 0}$ into a Hopf k-algebra, called the quantum double of $(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0}, \varphi)$. For simplicity, we write D_q instead of $D(\mathrm{U}_q^{\geq 0}, \mathrm{U}_q^{\leq 0})$.

Theorem 5.1. *As a k-algebra, D_q can be presented by the generators*

$$E_i, F_i, K_i, K_i^{-1}, \widetilde{K}_i, \widetilde{K}_i^{-1},\ (1 \leq i \leq n),$$

and the following relations:

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1 = K_i^{-1}K_i,$$
$$\widetilde{K}_i\widetilde{K}_j = \widetilde{K}_j\widetilde{K}_i, \quad \widetilde{K}_i\widetilde{K}_i^{-1} = 1 = \widetilde{K}_i^{-1}\widetilde{K}_i, \quad K_i\widetilde{K}_j = \widetilde{K}_jK_i,$$
$$K_iE_j = q^{d_ia_{i,j}}E_jK_i,\ K_iF_j = q^{-d_ia_{i,j}}F_jK_i,$$
$$\widetilde{K}_iE_j = q^{d_ia_{i,j}}E_j\widetilde{K}_i,\ \widetilde{K}_iF_j = q^{-d_ia_{i,j}}F_j\widetilde{K}_i,$$
$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - \widetilde{K}_i^{-1}}{q^{d_i} - q^{-d_i}},$$

$$\sum_{r+s=1-a_{i,j}} (-1)^s \begin{bmatrix} 1 - a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} E_i^rE_jE_i^s = 0, \quad \textit{if } i \neq j,$$
$$\sum_{r+s=1-a_{i,j}} (-1)^s \begin{bmatrix} 1 - a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} F_i^rF_jF_i^s = 0, \quad \textit{if } i \neq j.$$

Let M be a D_q-module such that $\mathrm{End}_{\mathrm{D}_q}(M)=k$. Note that the elements $K_i\widetilde{K}_i^{-1}, i=1,2,\cdots,n$, are invertible central elements in D_q. Therefore, there is a vector $\vec{z}=(z_1,\cdots,z_n)\in(k^\times)^n$, such that for every $1\leq i\leq n$, $K_i\widetilde{K}_i^{-1}$ acts as the scalar z_i on M. For each $0\neq z\in k$, we fix a square root $z^{1/2}$ of z. Let $\pi_{\vec{z}}^+$ be the k-algebra homomorphism $\mathrm{D}_q\to\mathrm{U}_q$ which is defined on generators by

$$\pi_{\vec{z}}^+(E_i)=z_i^{1/2}E_i,\ \pi_{\vec{z}}^+(F_i)=F_i,\ \pi_{\vec{z}}^+(K_i)=z_i^{1/2}K_i,\ \pi_{\vec{z}}^+(\widetilde{K}_i)=z_i^{-1/2}K_i,$$

for every $1\leq i\leq n$. It is easy to check that $\pi_{\vec{z}}^+$ is well-defined. Moreover, the kernel of $\pi_{\vec{z}}^+$, which is the ideal generated by $K_i\widetilde{K}_i^{-1}-z_i, i=1,2,\cdots,n$, annihilates the module M. It follows that M naturally becomes a module over the algebra U_q. Note that $\pi_{\vec{z}}^+$ is in general not a Hopf algebra map unless $\vec{z}=(1,1,\cdots,1)$.

We call a D_q-module M a weight D_q-module if $K_1,\cdots,K_n,\widetilde{K}_1,\cdots,\widetilde{K}_n$ all act semisimply on M. In that case, each $K_i\widetilde{K}_i^{-1}$ acts semisimply on M as well. Similarly, we call a U_q-module N a weight U_q-module if $K_1,K_2,\cdots,K_n$ all act semisimply on N.

Lemma 5.2. *Every finite dimensional simple (resp. indecomposable weight) D_q-module is the pull-back of a finite dimensional simple (resp. indecomposable weight) U_q-module through the algebra homomorphism $\pi_{\vec{z}}^+$ for some $\vec{z}=(z_1,\cdots,z_n)\in(k^\times)^n$.*

Let M be a U_q-module. Let $\vec{z}=(z_1,\cdots,z_n)\in(k^\times)^n$. We use $M_{\vec{z}}^+$ to denote the pull-back of M through the algebra homomorphism $\pi_{\vec{z}}^+$. Let $\varepsilon_{\vec{z}}^+$ be the one-dimensional representation of D_q which is defined on generators by

$$\varepsilon_{\vec{z}}^+(E_i)=0=\varepsilon_{\vec{z}}^+(F_i),\ \varepsilon_{\vec{z}}^+(K_i)=z_i^{1/2},\ \varepsilon_{\vec{z}}^+(\widetilde{K}_i)=z_i^{-1/2},\ i=1,2,\cdots,n.$$

It is easy to check that $\varepsilon_{\vec{z}}^+$ is well-defined.

Theorem 5.3. *The category $\widetilde{\mathcal{C}}$ of finite dimensional weight D_q-modules is equivalent to a direct sum of $|(k^\times)^n|$ copies of the category $\mathcal{C}$ of finite dimensional weight U_q-modules.*

With those one dimensional representations $\varepsilon_{\vec{z}}^+$ in mind, one can also formulate a version of Lemma 3.3 in the context of arbitrary Cartan matrix. As before, this provides an easy solution to the problem of decomposing the tensor product of certain D_q-modules, i.e., reducing them to the corresponding problem for U_q-modules.

ACKNOWLEDGMENTS

The first author would like to thank the School of Mathematics, Statistics and Computer Science, Victoria University of Wellington for their hospitality during his visit in 2005. He is grateful to the URF of VUW and

the Program NCET as well as NSFC (Project 10401005) for the financial support. The second author is supported by the Marsden Fund.

References

[1] H. H. Anderson, P. Polo and K.-x. Wen, Representations of quantum algebras, Invent. Math., **104**, 1–59, (1991).

[2] H.X. Chen, Irreducible representations of a class of quantum doubles, J. Algebra, **225**, 391–409, (2000).

[3] H.X. Chen, Finite-dimensional representations of a quantum double, J. Algebra, **251**, 751–789, (2002).

[4] C.De Concini and V.G. Kac, Representations of quantum groups at roots of 1, in: "Operator algebras, unitary representations, enveloping algebras, and invariant theory" (Paris, 1989), 471–506, Progr. Math., **92**, Birkhäuser Boston, Boston, MA, 1990.

[5] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycles–the quantum version, Comm. Algebra, **22**, 5715–5732, (1994).

[6] V.G. Drinfel'd, Hopf algebras and the quantum Yang-Baxter equations, Soviet math. Dokl., **32**, 254–258, (1985).

[7] V.G. Drinfel'd, On almost cocomutative Hopf algebras, Leningrad Math. J., **1**, 321–342, (1990).

[8] J.C. Jantzen, Lectures on quantum groups, Graduate Studies in Mathematics, **6**, American Mathematical Society, Providence, RI, 1996.

[9] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, Lett. Math. Physics, **10**, 63–69, (1985).

[10] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebras, and Yang-Baxter equation, Lett. Math. Physics, **11**, 247–252, (1986).

[11] A. Joseph, Quantum groups and their primitive ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **29**, Springer-Verlag, Berlin, 1995.

[12] C. Kassel, Quantum groups, Springer, Graduate Texts in Mathematics, **155**, New York, 1995.

[13] G. Lusztig, Modular representations and quantum groups, in: "Classical groups and related topics" (Beijing, 1987), 59–77, Contemp. Math., **82**, Amer. Math. Soc., Providence, RI, 1989.

[14] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. in Math., (2) **70**, 237–249, (1988).

[15] G. Lusztig, Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra, J. Amer. Math. Soc. (1) **3**, 257–296, (1990).

[16] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata, **35**, 89–113, (1990).

[17] G. Lusztig, Introduction to quantum groups, Progress in Math. **110** Birkhäuser, Boston, 1990.

[18] S. Majid, Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra, **130**, 17–64, (1990).

[19] M.K. Patra, On the structure of Nonsemisimple Hopf algebras, J. Phys. A. Math. Gen., **32**, 159–166, (1999).

[20] D.E. Radford, On oriented quantum algebras derived from representations of the quantum double of a finite-dimensional Hopf algebras, J. Algebra, **270**, 670–695, (2003).

[21] D.E. Radford and H-J. Schneider, Representations Parameterized by a Pair of Characters, preprint, ArXiv: math.QA/0603270, (2006).

[22] D.E. Radford and H-J. Schneider, On the irreducible representations of a class of pointed Hopf algebras, ArXiv: math.QA/0603269, (2006).

[23] R. Suter, Modules for $\mathfrak{U}_q(\mathfrak{sl}_2)$, Comm. Math. Phys., **163**, 359–393, (1994).
[24] E.J. Taft, The order of the antipode of a finite-dimensional Hopf algebras, Proc. Nat. Acad. Sci. U.S.A., **68**, 2631–2633, (1971).
[25] J. Xiao, Drinfel'd double and Ringel-Green theory of Hall algebras, J. Algebra, **190**, 100–144, (1997).
[26] J. Xiao, Finite dimensional representations of $U_t(sl_2)$ at roots of unity, Can. J. Math., (4) **49**, 772–787, (1997).

Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, P.R. China

E-mail address: `junhu303@yahoo.com.cn`

School of Mathematics, Statistics and Computer Science, Victoria University of Wellington, PO Box 600, Wellington, New Zealand

E-mail address: `yinhuo.zhang@vuw.ac.nz`